Implied Trinomial Trees

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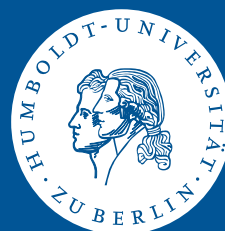
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Pavel Čížek and Karel Komorád

Options are financial derivatives that, conditional on the price of an underlying asset, constitute a right to transfer the ownership of this underlying. More specifically, a *European call* and *put options* give their owner the right to buy and sell, respectively, at a fixed strike price at a given date. Options are important financial instruments used for hedging since they can be included into a portfolio to reduce risk. Corporate securities (e.g., bonds or stocks) may include option features as well. Last, but not least, some new financing techniques, such as contingent value rights, are straightforward applications of options. Thus, option pricing has become one of the basic techniques in finance.

The boom in research on the use of options started after Black and Scholes (1973) published an option-pricing formula based on geometric Brownian motion. Option prices computed by the Black-Scholes formula and the market prices of options exhibit a discrepancy though. Whereas the volatility of market option prices varies with the price (or moneyness) – the dependency referred to as the *volatility smile*, the Black-Scholes model is based on the assumption of a constant volatility. Therefore, to model option prices consistent with the market many new approaches were proposed. Probably the most commonly used and rather intuitive procedure for option pricing is based on binomial trees, which represent a discrete form of the Black-Scholes model. To fit the market data, Derman and Kani (1994) proposed an extension of binomial trees: the so-called implied binomial trees, which are able to model the market volatility smile.

Implied trinomial trees (ITTs) present an analogous extension of trinomial trees proposed by Derman, Kani, and Chriss (1996). Like their binomial counterparts, they can fit the market volatility smile and actually converge to the same continuous limit as binomial trees. In addition, they allow for a free choice of the underlying prices at each node of a tree, the so-called state space. This feature of ITTs allows to improve the fit of the volatility smile under some circumstances such as inconsistent, arbitrage-violating, or other market prices leading to implausible or degenerated probability distributions in binomial trees. We introduce ITTs in several steps. We first review main concepts regarding option pricing (Section 1) and implied models (Section 2). Later, we discuss the construction of ITTs (Section 3) and provide some illustrative examples (Section 4).

1 Option Pricing

The option-pricing model by Black and Scholes (1973) is based on the assumptions that the underlying asset follows a geometric Brownian motion with a constant volatility σ:

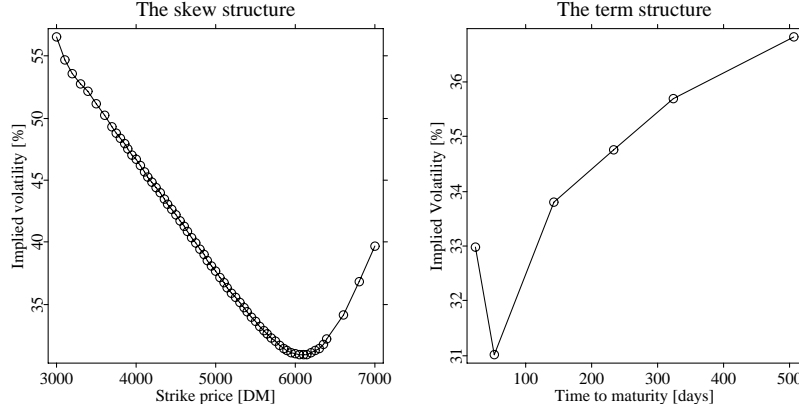$$\frac{dS_t}{S_t} \;\;=\;\; \mu dt + \sigma dW_t, \tag{1}$$

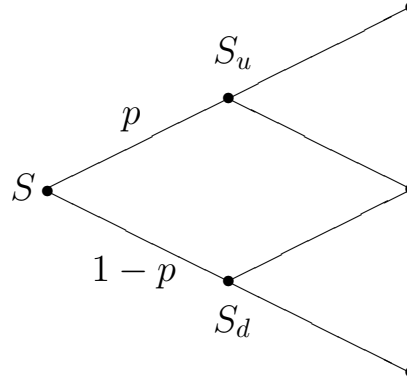Figure 1: Implied volatilities of DAX put options on January 29, 1999.

Figure 2: Two levels of a CRR binomial tree.

where S_t denotes the underlying-price process, μ is the expected return, and W_t stands for the standard Wiener process. As a consequence, the distribution of S_t is lognormal. More importantly, the volatility σ is the only parameter of the Black-Scholes formula which is not explicitly observable on the market. Thus, we infer on σ by matching the observed option prices. A solution σ_I, "implied" by options prices, is called the *implied volatility* (or Black-Scholes equivalent). In general, implied volatilities vary both with respect to the exercise price (the skew structure) and expiration time (the term structure). Both dependencies are illustrated in Figure 1, with the first one representing the volatility smile. Let us add that the implied volatility of an option is the market's estimate of the average future underlying volatility during the life of that option. We refer to the market's estimate of an underlying volatility at a particular time and price point as the *local volatility*.

2

Binomial trees, as a discretization of the Black-Scholes model, can be constructed in several alternative ways. Here we recall the classic Cox, Ross, and Rubinstein's (1979) scheme (CRR), which has a constant logarithmic spacing between nodes on the same level (this spacing represents the future price volatility). A standard CRR tree is depicted in Figure 2. Starting at a node S, the price of an underlying asset can either increase to S_u with probability p or decrease to S_d with probability $1 - p$:

$$S_u = Se^{\sigma\sqrt{\Delta t}}, \tag{2}$$

$$S_d = Se^{-\sigma\sqrt{\Delta t}}, \tag{3}$$

$$p = \frac{F - S_d}{S_u - S_d}, \tag{4}$$

where Δt refers to the time step and σ is the (constant) volatility. The forward price $F = e^{r\Delta t}S$ in the node S is determined by the the continuous interest rate r (for the sake of simplicity, we assume that the dividend yield equals zero; see Cox, Ross, and Rubinstein, 1979, for treatment of dividends).

A binomial tree corresponding to the risk-neutral underlying evaluation process is the same for all options on this asset, no matter what the strike price or time to expiration is. There are many extensions of the original Black-Scholes approach that try to capture the volatility variation and to price options consistently with the market prices (that is, to account for the volatility smile). Some extensions incorporate a stochastic volatility factor or discontinuous jumps in the underlying price; see for instance Franke, Härdle, and Hafner (2004).In the next section, we discuss an extension of the Black-Scholes model developed by Derman and Kani (1994) – the implied trees.

2   Trees and Implied Trees

While the Black-Scholes model assumes that an underlying asset follows a geometric Brownian motion (1) with a constant volatility, more complex models assume that the underlying follows a process with a price- and time-varying volatility $\sigma(S, t)$. See Dupire (1994) and Fengler, Härdle, and Villa (2003) for details and related evidence. Such a process can be expressed by the following stochastic differential equation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma(S, t)dW_t. \tag{5}$$

This approach ensures that the valuation of an option remains preference-free, that is, all uncertainty is in the spot price, and thus, we can hedge options using the underlying.

Derman and Kani (1994) show that it is possible to determine $\sigma(S, t)$ directly from the market prices of liquidly traded options. Further, they use this volatility $\sigma(S, t)$ to construct an implied binomial tree (IBT), which is a natural discrete representation of a non-lognormal evolution process of the underlying prices. In general, we can use – instead of an IBT – any (higher-order) multinomial tree for the discretization of process (5). Nevertheless, as the time step
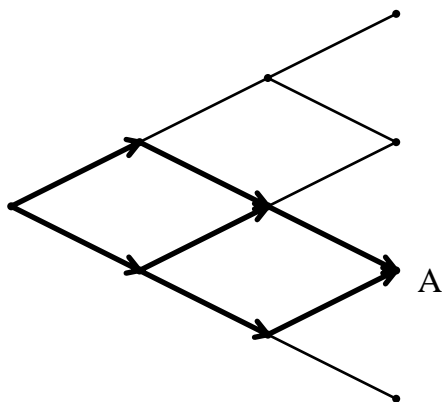
Figure 3: Computing the Arrow-Debreu price in a binomial tree. The bold lines with arrows depict all (three) possible path from the root of the tree to point A.

tends towards zero, all of them converge to the same continuous process (Hull and White, 1990). Thus, IBTs are among all implied multinomial trees minimal in the sense that they have only one degree of freedom – the arbitrary choice of the central node at each level of the tree. Although one may feel now that binomial trees are sufficient, some higher-order trees could be more useful because they allow for a more flexible discretization in the sense that transition probabilities and probability distributions can vary as smoothly as possible across a tree. This is especially important when the market option prices are inaccurate because of inefficiency, market frictions, and so on.

At the end of this section, let us to recall the concept of Arrow-Debreu prices, which is closely related to multinomial trees and becomes very useful in subsequent derivations (Section 3). Let (n, i) denote the ith (highest) node in the nth time level of a tree. The Arrow-Debreu price $\lambda_{n,i}$ at node (n, i) of a tree is computed as the sum of the product of the risklessly discounted transition probabilities over all paths starting in the root of the tree and leading to node (n, i). Hence, the Arrow-Debreu price of the root is equal to one and the Arrow-Debreu prices at the final level of a (multinomial) tree form a discrete approximation of the state price density. Notice that these prices are discounted, and thus, the risk-neutral probability corresponding to each node (at the final level) should be calculated as the product of the Arrow-Debreu price and the capitalizing factor e^{rT}.

3   Implied Trinomial Trees

3.1   Basic Insight

A trinomial tree with N levels is a set of nodes $s_{n,i}$ (representing the underlying price), where $n = 1, \ldots, N$ is the level number and $i = 1, \ldots, 2n-1$ indexes nodes within a level. Being at a node $s_{n,i}$, one can move to one of three nodes (see Figure 4a): (i) to the upper node with value $s_{n+1,i}$ with probability p_i; (ii) to the lower node with value $s_{n+1,i+2}$ with probability q_i; and (iii) to the middle node with value $s_{n+1,i+1}$ with probability $1 - p_i - q_i$. For the sake of brevity, we omit the level index n from transition probabilities unless they refer to a specific level; that is, we write p_i and q_i instead of $p_{n,i}$ and $q_{n,i}$ unless the level has to be specified. Similarly, let us denote the nodes in the new level with capital letters: $S_i(=s_{n+1,i})$, $S_{i+1}(=s_{n+1,i+1})$ and $S_{i+2}(=s_{n+1,i+2})$, respectively (see Figure 4b).
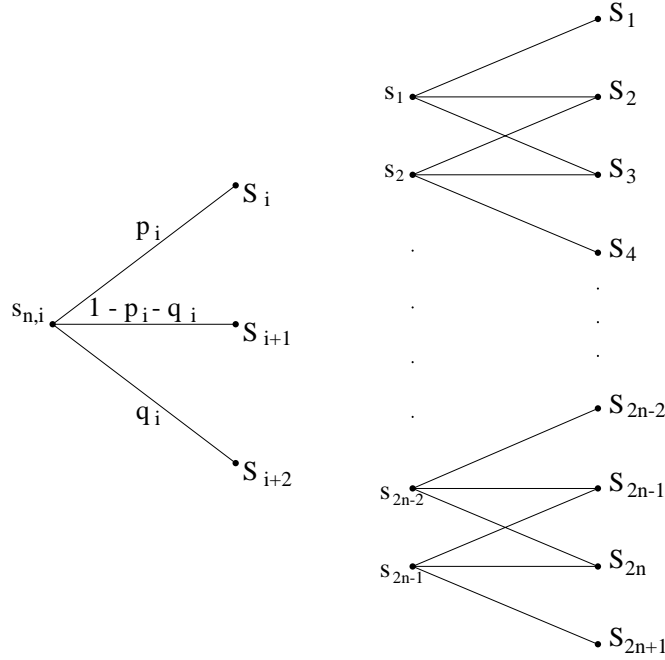

Figure 4: Nodes in a trinomial tree. *Left panel:* a single node with its branches. *Right panel:* the nodes of two consecutive levels $n-1$ and n.

Starting from a node $s_{n,i}$ at time t_n, there are five unknown parameters: two transition probabilities p_i and q_i and three prices S_i, S_{i+1}, and S_{i+2} at new nodes. To determine them, we need to introduce the notation and main requirements a tree should satisfy. First, let F_i denote the known forward price of the spot price $s_{n,i}$ and $\lambda_{n,i}$ the known Arrow-Debreu price at node (n, i). The Arrow-Debreu prices for a trinomial tree can be obtained by the following

iterative formulas:

$$\lambda_{1,1} = 1, \qquad (6)$$

$$\lambda_{n+1,1} = e^{-r\Delta t}\lambda_{n,1}p_1, \qquad (7)$$

$$\lambda_{n+1,2} = e^{-r\Delta t}\{\lambda_{n,1}(1-p_1-q_1)+\lambda_{n,2}p_2\}, \qquad (8)$$

$$\lambda_{n+1,i+1} = e^{-r\Delta t}\{\lambda_{n,i-1}q_{i-1}+\lambda_{n,i}(1-p_i-q_i)+\lambda_{n,i+1}p_{i+1}\}, \qquad (9)$$

$$\lambda_{n+1,2n} = e^{-r\Delta t}\{\lambda_{n,2n-1}(1-p_{2n-1}-q_{2n-1})+\lambda_{n,2n-2}q_{2n-2}\}, \qquad (10)$$

$$\lambda_{n+1,2n+1} = e^{-r\Delta t}\lambda_{n,2n-1}q_{2n-1}. \qquad (11)$$

An implied tree provides a discrete representation of the evolution process of underlying prices. To capture and model the underlying price correctly, we desire that an implied tree:

1. reproduces correctly the volatility smile,

2. is risk-neutral,

3. uses transition probabilities from interval $(0,1)$.

To fulfill the risk-neutrality condition, the expected value of the underlying price $s_{n+1,i}$ in the following time period t_{n+1} has to equal its known forward price:

$$\mathrm{E}s_{n+1,i} = p_iS_i + (1-p_i-q_i)S_{i+1} + q_iS_{i+2} = F_i = e^{r\Delta t}s_{n,i}, \qquad (12)$$

where r denotes the continuous interest rate and Δt is the time step from t_n to t_{n+1}. Additionally, one can specify such a condition also for the second moments of $s_{n,i}$ and F_i. Hence, one obtains a second constraint on the node prices and transition probabilities:

$$p_i(S_i - F_i)^2 + (1-p_i-q_i)(S_{i+1}-F_i)^2 + q_i(S_{i+2}-F_i)^2 = F_i^2\sigma_i^2\Delta t + o(\Delta t), \qquad (13)$$

where σ_i is the stock or index price volatility during the time period.

Consequently, we have two constraints (12) and (13) for five unknown parameters, and therefore, there is no unique implied trinomial tree. On the other hand, all trees satisfying these constraints are equivalent in the sense that as the time spacing Δt tends to zero, all these trees converge to the same continous process. A common method for constructing an ITT is to choose first freely the underlying prices and then to solve equations (12) and (13) for the transition probabilities p_i and q_i. Afterwards one only has to ensure that these probabilities do not violate the above mentioned Condition 3. Apparently, using an ITT instead of an IBT gives us additional degrees of freedom. This allows us to better fit the volatility smile, especially when inconsistent or arbitrage-violating market option prices make a consistent tree impossible. Note, however, that even though the constructed tree is consistent, other difficulties can arise when its local volatility and probability distributions are jagged and "implausible."

3.2 State Space

There are several methods we can use to construct an initial state space. Let us first discuss a construction of a constant-volatility trinomial tree, which forms
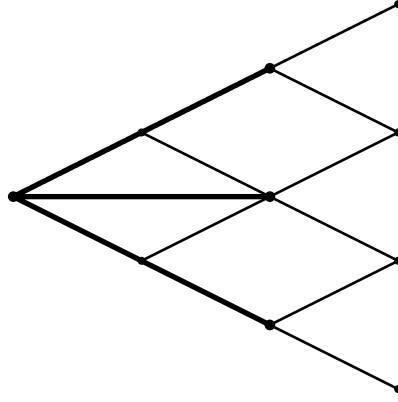
Figure 5: Constructing a constant-volatility trinomial tree (thick lines) by combining two steps of a CRR binomial tree (thin lines).

a base for an implied trinomial tree. As already mentioned, binomial and trinomial discretization of the constant-volatility Black-Scholes model have the same continous limit, and therefore, are equivalent. Hence, we can start from a constant-volatility CRR binomial tree and then combine two steps of this tree into a single step of a new trinomial tree. This is illustrated in Figure 5, where thin lines correspond to the original binomial tree and the thick lines to the constructed trinomial tree.

Consequently, using formulas (2) and (3), we can derive the following expressions for the nodes of the constructed trinomial tree:

$$S_i \quad = s_{n+1,i} \quad = s_{n,i} \; e^{\sigma\sqrt{2\Delta t}}, \tag{14}$$

$$S_{i+1} \quad = s_{n+1,i+1} \quad = s_{n,i}, \tag{15}$$

$$S_{i+2} \quad = s_{n+1,i+2} \quad = s_{n,i} \; e^{-\sigma\sqrt{2\Delta t}}, \tag{16}$$

where σ is a constant volatility (e.g., an estimate of the at-the-money volatility at maturity T). Next, summing the transition probabilities in the binomial tree given in (4), we can also derive the up and down transition probabilities in the trinomial tree (the "middle" transition probability is equal to $1 - p_i - q_i$):

$$p_i \quad = \quad \left(\frac{e^{r\Delta t/2} - e^{-\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}} \right)^2,$$

$$q_i \quad = \quad \left(\frac{e^{\sigma\sqrt{\Delta t/2}} - e^{r\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}} \right)^2.$$

Note that there are more methods for building a constant-volatility trinomial tree such as combining two steps of a Jarrow and Rudd's (1983) binomial tree; see Derman, Kani, and Chriss (1996) for more details.

When the implied volatility varies only slowly with strike and expiration, the regular state space with a uniform mesh size, as described above, is adequate for constructing ITT models. On the other hand, if the volatility varies significantly with strike or time to maturity, we should choose a state space reflecting these properties. Assuming that the volatility is separable in time and stock price, $\sigma(S,t) = \sigma(S)\sigma(t)$, an ITT state space with a proper skew and term structure can be constructed in four steps.

First, we build a regular trinomial lattice with a constant time spacing Δt and a constant price spacing ΔS as described above. Additionally, we assume that all interest rates and dividends are equal to zero.

Second, we modify Δt at different time points. Let us denote the original equally spaced time points $t_0 = 0, t_1, \ldots, t_n = T$. We can then find the unknown scaled times $\tilde{t}_0 = 0, \tilde{t}_1, \ldots, \tilde{t}_n = T$ by solving the following set of non-linear equations:

$$\tilde{t}_k \sum_{i=1}^{n-1} \frac{1}{\sigma^2(\tilde{t}_i)} + \tilde{t}_k \frac{1}{\sigma^2(T)} = T \sum_{i=1}^{k} \frac{1}{\sigma^2(\tilde{t}_i)}, \qquad k = 1, \ldots, n-1. \qquad (17)$$

Next, we change ΔS at different levels. Denoting by $S_1, \ldots, S_{2n+1}$ the original (known) underlying prices, we solve for rescaled underlying prices $\tilde{S}_1, \ldots, \tilde{S}_{2n+1}$ using

$$\frac{\tilde{S}_k}{\tilde{S}_{k-1}} = \exp\left\{\frac{c}{\sigma(S_k)} \ln \frac{S_k}{S_{k-1}}\right\}, \qquad k = 2, \ldots, 2n+1, \qquad (18)$$

where c is a constant. It is recommended to set c to an estimate of the local volatility. Since there are $2n$ equations for $2n + 1$ unknown parameters, an additional equation is needed. Here we always suppose that the new central node equals the original central node: $\tilde{S}_{n+1} = S_{n+1}$. See Derman, Kani, and Chriss (1996) for a more elaborate explanation of the theory behind equations (17) and (18).

Finally, one can increase all node prices by a sufficiently large growth factor, which removes forward prices violations, see Section 3.4. Multiplying all zero-rate node prices at time $\tilde{t}_i$ by $e^{r\tilde{t}_i}$ should be always sufficient.

3.3 Transition Probabilities

Once the state space of an ITT is fixed, we can compute the transition probabilities for all nodes (n,i) at each tree level n. Let $C(K, t_{n+1})$ and $P(K, t_{n+1})$ denote today's price of a standard European call and put option, respectively, struck at K and expiring at t_{n+1}. These values can be obtained by interpolating the smile surface at various strike and time points. The values of these options given by the trinomial tree are the discounted expectations of the pay-off functions: $\max(S_j - K, 0) = (S_j - K)^+$ for the call option and $\max(K - S_j, 0)$ for the put option at the node $(n+1, j)$. The expectation is taken with respect to the probabilities of reaching each node, that is, with respect to transition

probabilities:

$$C\left(K, t_{n+1}\right) \quad = \quad e^{-r\Delta t} \sum_j \{p_j \lambda_{n,j} + (1 - p_{j-1} - q_{j-1})\lambda_{n,j-1} \quad (19)$$

$$+ q_{j-2}\lambda_{n,j-2}\} \left(S_j - K\right)^+,$$

$$P\left(K, t_{n+1}\right) \quad = \quad e^{-r\Delta t} \sum_j \{p_j \lambda_{n,j} + (1 - p_{j-1} - q_{j-1})\lambda_{n,j-1} \quad (20)$$

$$+ q_{j-2}\lambda_{n,j-2}\} \left(K - S_j\right)^+.$$

If we set the strike price K to S_{i+1} (the stock price at node $(n+1, i+1)$), rearrange the terms in the sum, and use equation (12), we can express the transition probabilities p_i and q_i for all nodes above the central node from formula (19):

$$p_i \quad = \quad \frac{e^{r\Delta t}C(S_{i+1}, t_{n+1}) - \sum_{j=1}^{i-1} \lambda_{n+1,j}(F_j - S_{i+1})}{\lambda_{n+1,i}(S_i - S_{i+1})}, \quad (21)$$

$$q_i \quad = \quad \frac{F_i - p_i(S_i - S_{i+1}) - S_{i+1}}{S_{i+2} - S_{i+1}}. \quad (22)$$

Similarly, we compute from formula (20) the transition probabilities for all nodes below (and including) the center node $(n+1, n)$ at time t_n:

$$q_i \quad = \quad \frac{e^{r\Delta t}P(S_{i+1}, t_{n+1}) - \sum_{j=i+1}^{2n-1} \lambda_{n+1,j}(S_{i+1} - F_j)}{\lambda_{n+1,i}(S_{i+1} - S_{i+2})}, \quad (23)$$

$$p_i \quad = \quad \frac{F_i - q_i(S_{i+2} - S_{i+1}) - S_{i+1}}{S_i - S_{i+1}}. \quad (24)$$

A detailed derivation of these formulas can be found in Komorád (2002). Finally, the implied local volatilities are approximated from equation (13):

$$\sigma_i^2 \approx \frac{p_i(S_i - F_i)^2 + (1 - p_i - q_i)(S_{i+1} - F_i)^2 + q_i(S_{i+2} - F_i)^2}{F_i^2 \Delta t}. \quad (25)$$

3.4 Possible Pitfalls

Formulas (21)–(24) can unfortunately result in transition probabilities which are negative or greater than one. This is inconsistent with rational option prices and allows arbitrage. We actually have to face two forms of this problem, see Figure 6 for examples of such trees. First, we have to check that no forward price $F_{n,i}$ at node (n, i) falls outside the range of its daughter nodes at the level $n + 1$: $F_{n,i} \in (s_{n+1,i+2}, s_{n+1,i})$. This inconsistency is not difficult to overcome since we are free to choose the state space. Thus, we can overwrite the nodes causing this problem.

Second, extremely small or large values of option prices, which would imply an extreme value of local volatility, can also result in probabilities that are negative or larger than one. In such a case, we have to overwrite the option prices which
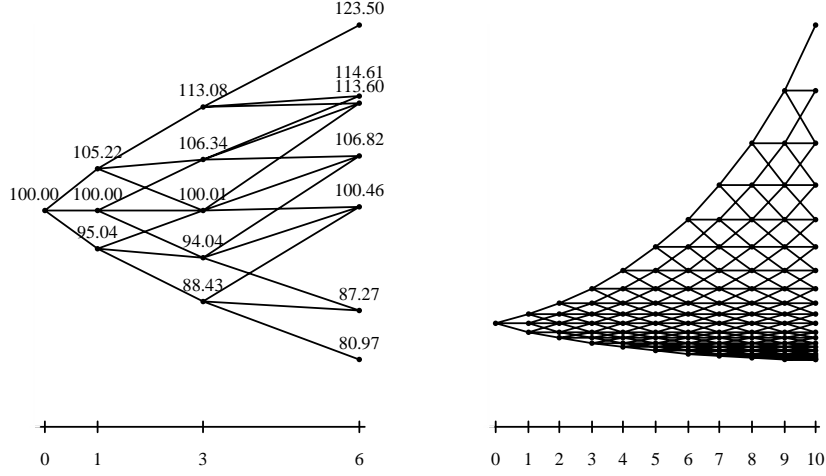
Figure 6: Two kinds of the forward price violation. *Left panel:* forward price outside the range of its daughter nodes. *Right panel:* sharp increase in option prices leading to an extreme local volatility.

led to the unacceptable probabilities. Fortunately, the transition probabilities can be always corrected providing that the corresponding state space does not violate the forward price condition $F_{n,i} \in (s_{n+1,i+2}, s_{n+1,i})$. Derman, Kani, and Chriss (1996) proposed to reduce the troublesome nodes to binomial ones or to set

$$p_i = \frac{1}{2} \left(\frac{F_i - S_{i+1}}{S_i - S_{i+1}} + \frac{F_i - S_{i+2}}{S_i - S_{i+2}} \right), \quad q_i = \frac{1}{2} \left(\frac{S_i - F_i}{S_i - S_{i+2}} \right), \qquad (26)$$

for $F_i \in (S_{i+1}, S_i)$ and

$$p_i = \frac{1}{2} \left(\frac{F_i - S_{i+2}}{S_i - S_{i+2}} \right), \quad q_i = \frac{1}{2} \left(\frac{S_{i+1} - F_i}{S_{i+1} - S_{i+2}} + \frac{S_i - F_i}{S_i - S_{i+2}} \right), \qquad (27)$$

for $F_i \in (S_{i+2}, S_{i+1})$. In both cases, the "middle" transition probability is equal to $1 - p_i - q_i$.

4   Examples

To illustrate the construction of an implied trinomial tree and its use, we consider here ITTs for two artificial implied-volatility functions and an implied-volatility function constructed from real data.

4.1   Pre-specified Implied Volatility

Let us consider a case where the volatility varies only slowly with respect to the strike price and time to expiration (maturity). Assume that the current
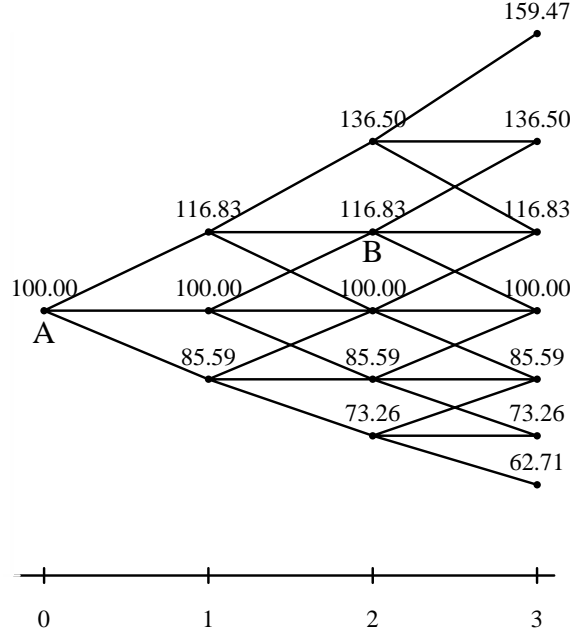
Figure 7: The state space of a trinomial tree with constant volatility $\sigma = 11\%$. Nodes A and B are reference points for which we demonstrate constructing of an ITT and estimating of the implied local volatility.

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index level is 100 points, the annual riskless interest rate is $r = 12\%$, and the dividend yield equals $\delta = 4\%$. The annualized Black-Scholes implied volatility is assumed to be $\sigma = 11\%$, and additionally, it increases (decreases) linearly by 10 basis points (i.e., 0.1%) with every 10 unit drop (rise) in the strike price K; that is, $\sigma_I = 0.11 - \Delta K * 0.001$. To keep the example simple, we consider three one-year steps.

First, we construct the state space: a constant-volatility trinomial tree as described in Section 3.2. The first node at time $t_0 = 0$, labeled A in Figure 7, has the value of $s_A = 100$, today's spot price. The next three nodes at time t_1, are computed from equations (14)–(16) and take values $S_1 = 116.83$, $S_2 = 100.00$, and $S_3 = 85.59$, respectively. In order to determine the transition probabilities, we need to know the price $P(S_2, t_1)$ of a put option struck at $S_2 = 100$ and expiring one year from now. Since the implied volatility of this option is 11%, we calculate its price using a constant-volatility trinomial tree with the same state space and find it to be 0.987 index points. Further, the forward price corresponding to node A is $F_A = Se^{(r^* - \delta^*)\Delta t} = 107.69$, where $r^* = \log(1 + r)$ denotes the continuous interest rate and $\delta^* = \log(1 + \delta)$ the continuous dividend rate. Hence, the transition probability of a down movement computed
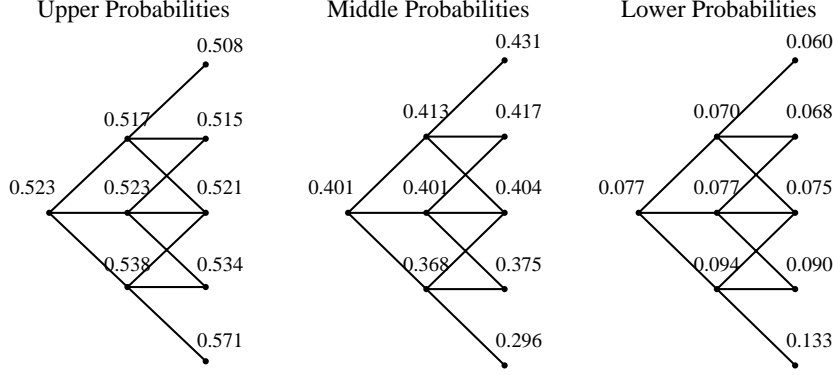
Figure 8: Transition probabilities for $\sigma_I = 0.11 - \Delta K \cdot 0.001$.

from equation (23) is

$$q_A = \frac{e^{\log(1+0.12)\cdot 1} 0.987 - \Sigma}{1 \cdot (100.00 - 85.59)} = 0.077,$$

where the summation term Σ in the numerator is zero because there are no nodes with price lower than S_3 at time t_1. Similarly, the transition probability of an upward movement p_A computed from equation (24) is

$$p_A = \frac{107.69 + 0.077 \cdot (100.00 - 85.59) - 100}{116.83 - 100.00} = 0.523.$$

Finally, the middle transition probability equals $1 - p_A - q_A = 0.4$. As one can see from equations (6)–(11), the Arrow-Debreu prices turn out to be just discounted transition probabilities: $\lambda_{1,1} = e^{-\log(1+0.12)\cdot 1} \cdot 0.523 = 0.467$, $\lambda_{1,2} = 0.358$, and $\lambda_{1,3} = 0.069$. Finally, we can estimate the value of the implied local volatility at node A from equation (25), obtaining $\sigma_A = 9.5\%$.

Let us demonstrate the computation of one further node. Starting from node B in year $t_2 = 2$ of Figure 7, the index level at this node is $s_B = 116.83$ and its forward price one year later is $F_B = e^{(r^* - \delta^*)\cdot 1} \cdot 116.83 = 125.82$. From this node, the underlying can move to one of three future nodes at time $t_3 = 3$, with prices $s_{3,2} = 136.50$, $s_{3,3} = 116.83$, and $s_{3,4} = 100.00$. The value of a call option struck at 116.83 and expiring at time $t_3 = 3$ is $C(s_{3,3}, t_3) = 8.87$, corresponding to the implied volatility of 10.83% interpolated from the smile. The Arrow-Debreu price computed from equation (8) is

$$\lambda_{2,2} = e^{-\log(1+0.12)\cdot 1}\{0.467 \cdot (1 - 0.517 - 0.070) + 0.358 \cdot 0.523\} = 0.339.$$

The numerical values used here are already known from the previous level at time t_1. Now, using equations (21) and (22) we can find the transition proba-
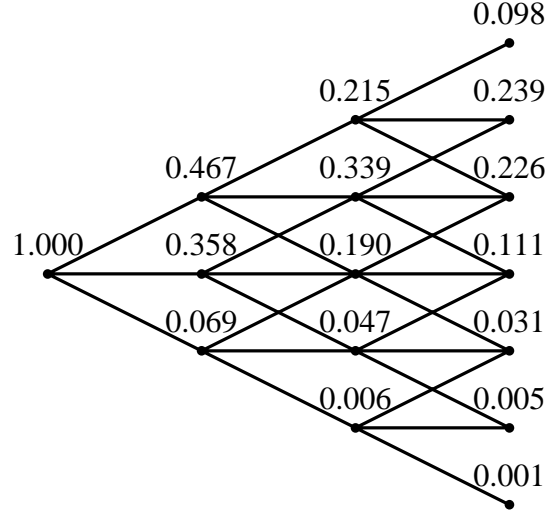
Figure 9: Arrow-Debreu prices for $\sigma_I = 0.11 - \Delta K \cdot 0.001$.

bilities:

$$
\begin{aligned}
p_{2,2} &= \frac{e^{\log(1+0.12)\cdot 1} \cdot 8.87 - \Sigma}{0.339 \cdot (136.50 - 116.83)} = 0.515, \\
q_{2,2} &= \frac{125.82 - 0.515 \cdot (136.50 - 116.83) - 116.83}{100 - 116.83} = 0.068,
\end{aligned}
$$

where Σ contributes only one term $0.215 \cdot (147 - 116.83)$, that is, there is one single node above S_B whose forward price is equal to 147. Finally, employing (25) again, we find that the implied local volatility at this node is $\sigma_B = 9.3\%$.

The complete trees of transition probabilities, Arrow-Debreu prices, and local volatilities for this example are shown in Figures 8–10.

As already mentioned in Section 3.4, the transition probabilities may fall out of the interval $(0, 1)$. For example, let us slightly modify our previous example and assume that the Black-Scholes volatility increases (decreases) linearly 0.5 percentage points with every 10 unit drop (rise) in the strike price K; that is, $\sigma_I = 0.11 - \Delta K \cdot 0.005$. In other words, the volatility smile is now five times steeper than before. Using the same state space as in the previous example, we find inadmissable transition probabilities at nodes C and D, see Figures 11–13. To overwrite them with plausible values, we used the strategy described by (26) and (27) and obtained reasonable results in the sense of the three conditions stated on page 6.
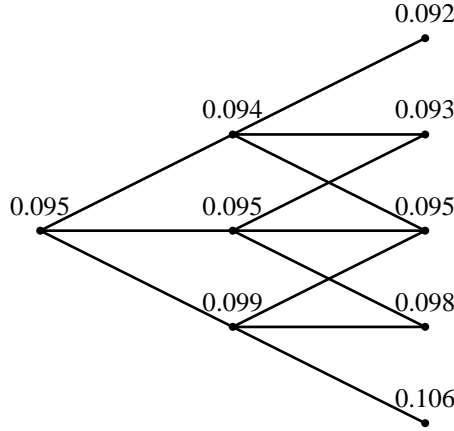
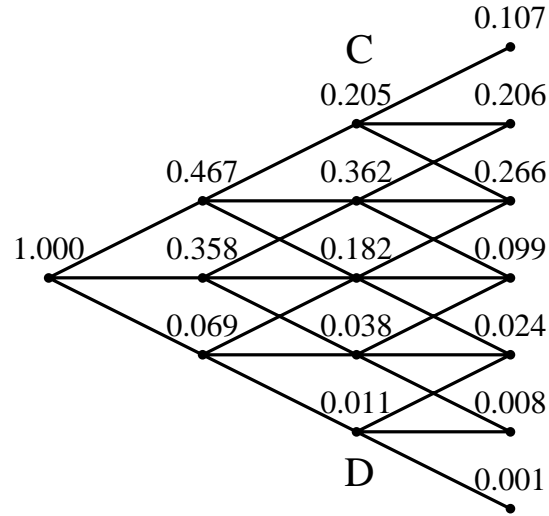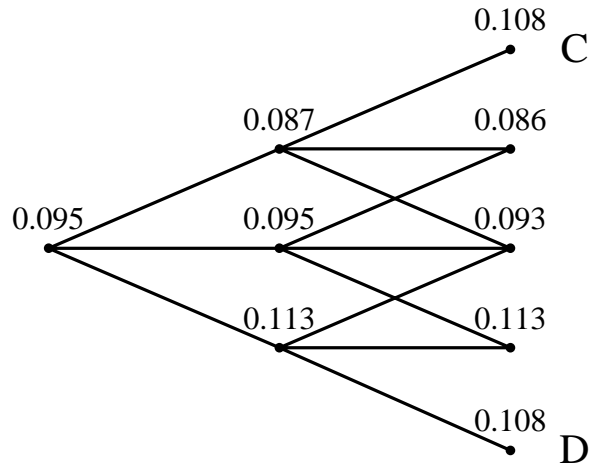Figure 10: Implied local volatilities for $\sigma_I = 0.11 - \Delta K \cdot 0.001$.
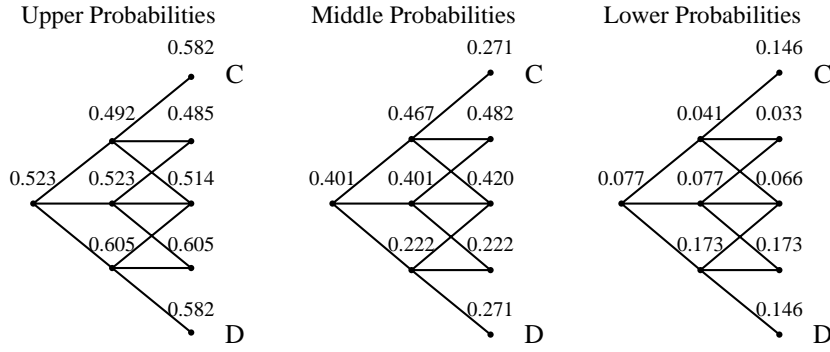
Figure 11: Transition probabilities for $\sigma_I = 0.11 - \Delta K \cdot 0.005$. Nodes C and D had inadmissible transition probabilities (21)–(24).

4.2 German Stock Index

Following the artificial examples, let us now demonstrate the ITT modeling for a real data set, which consists of strike prices for DAX options with maturities from two weeks to two months on January 4, 1999. Given such data, we can firstly compute from the Black-Scholes equation (1) the implied volatilities at various combinations of prices and maturities, that is, we can construct the volatility smile. Next, we build and calibrate an ITT so that it fits this smile. The procedure is analogous to the examples described above – the only difference lies in replacing an artificial function $\sigma_I(K, t)$ by an estimate of implied volatility σ_I at each point (K, t).

Figure 12: Arrow-Debreu prices for $\sigma_I = 0.11 - \Delta K \cdot 0.005$. Nodes C and D had inadmissible transition probabilities (21)–(24).

STFitt06.xpl

Figure 13: Implied local volatilities for $\sigma_I = 0.11 - \Delta K \cdot 0.005$. Nodes C and D had inadmissible transition probabilities (21)–(24).

STFitt07.xpl

For the purpose of demonstration, we build a three-level ITT with time step
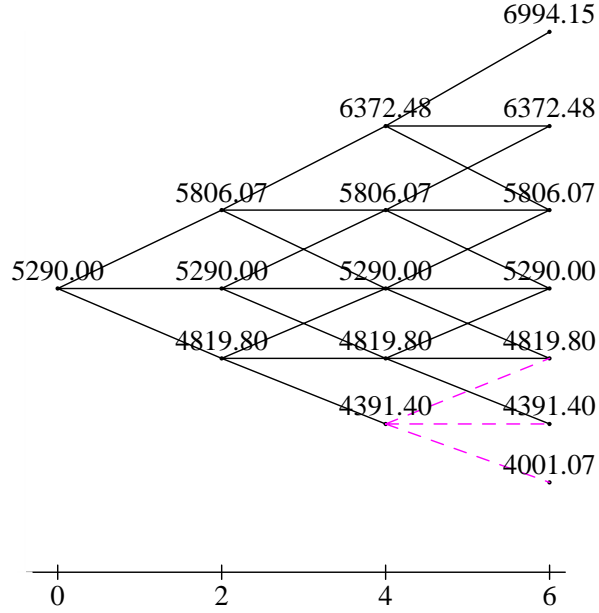
Figure 14: The state space of the ITT constructed for DAX on January 4, 1999. Dashed lines mark the transitions with originally inadmissible transition probabilities.

Δt of two weeks. First, we construct the state space (Section 3.2) starting at time $t_0 = 0$ with the spot price $S = 5290$ and riskless interest rate $r = 4\%$, see Figure 14. Further, we have to compute the transition probabilities. Because option contracts are not available for each combination of price and maturity, we use a nonparametric smoothing procedure to model the whole volatility surface $\sigma_I(K, t)$ as employed by Aït-Sahalia, Wang, and Yared (2001) and Fengler, Härdle, and Villa (2003), for instance. Given the data, some transition probabilities fall outside interval $(0, 1)$; they are depicted by dashed lines in Figure 14. Such probabilities have to be corrected as described in Section 3.4 (there are no forward price violations). The resulting local volatilities, which reflect the volatility skew, are on Figure 15.

Probably the main result of this ITT model can be summarized by the state price density (the left panel of Figure 16). This density describes the price distribution given by the constructed ITT and smoothed by the Nadaraya-Watson estimator. Apparently, the estimated density is rather rough because we used just three steps in our tree. To get a smoother state-price density estimate, we doubled the number of steps; that is, we used six one-week steps instead of three two-week steps (see the right panel of Figure 16).

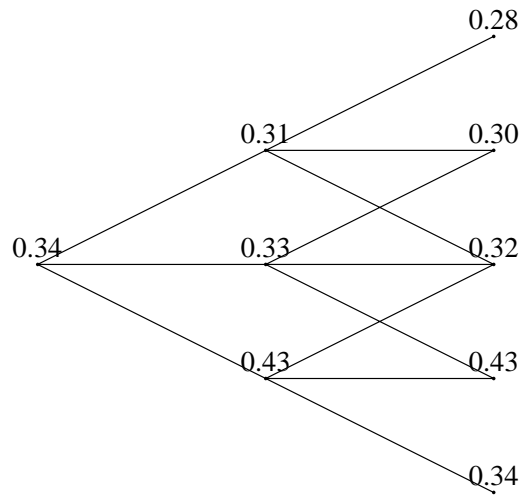Finally, it possible to use the constructed ITT to evaluate various DAX options.

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Figure 15: Implied local volatilities computed from an ITT for DAX on January 4, 1999.
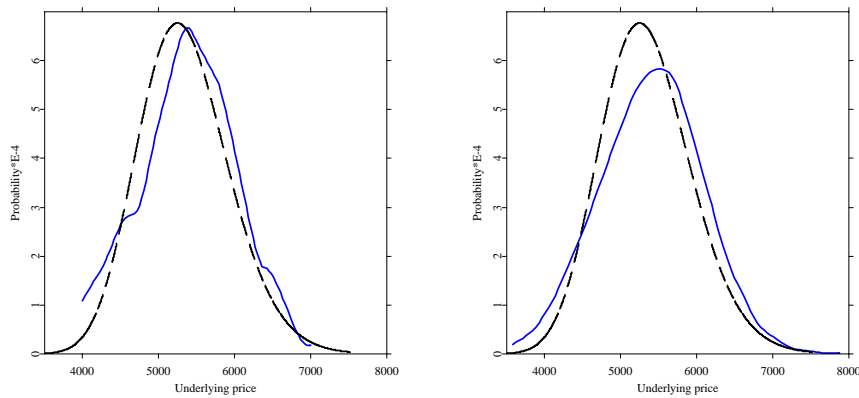
STFitt08.xpl

Figure 16: State price density estimated from an ITT for DAX on January 4, 1999. The dashed line depicts the corresponding Black-Scholes density. *Left panel:* State price density for a three-level tree. *Right panel:* State price density for a six-level tree.

STFitt08.xpl

STFitt09.xpl

For example, a European knock-out option gives the owner the same rights as a standard European option as long as the index price S does not exceed
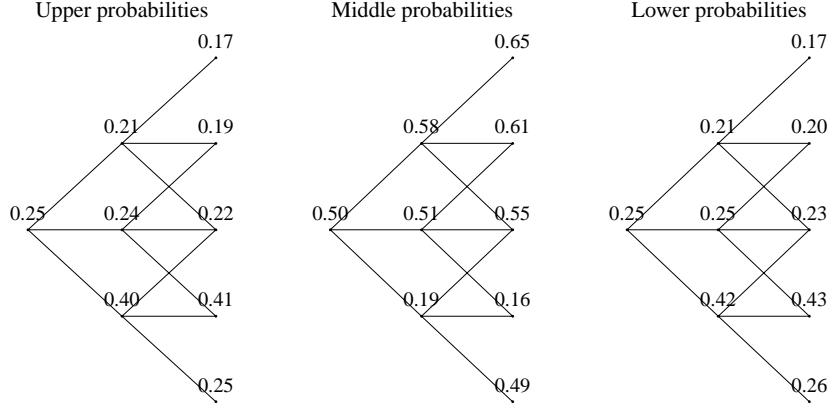
Figure 17: Transition probabilities of the ITT constructed for DAX on January 4, 1999.

STFitt10.xpl

or fall below some barrier B for the entire life of the knock-out option; see Härdle, Kleinow, and Stahl (2002) for details. So, let us compute the price of the knock-out-call DAX option at maturity $T = 6$ weeks, strike price $K = 5200$, and barrier $B = 4800$. The option price at time t_j ($t_0 = 0, t_1 = 2, t_2 = 4$, and $t_3 = 6$ weeks) and stock price $s_{j,i}$ will be denoted $V_{j,i}$.

At the maturity $t = T = 6$, the price is known: $V_{3,i} = \max\{0, s_{3,i} - K\}, i = 1, \ldots, 7$. Thus, $V_{3,1} = \max\{0, 4001.01 - 5200\} = 0$ and $V_{3,5} = \max\{0, 5806.07 - 5200\} = 606.07$, for instance. To compute the option price at $t_j < T$, one just has to discount the conditional expectation of the option price at time t_{j+1}

$$V_{j,i} = e^{-r^* \Delta t}\{p_{j,i}V_{j+1,i+2} + (1 - p_{j,i} - q_{j,i})V_{j+1,i+1} + q_{j,i}V_{j+1,i}\} \qquad (28)$$

provided that $s_{j,i} \geq B$, otherwise $V_{j,i} = 0$. Hence at time $t_2 = 4$, one obtains $V_{2,1} = 0$ because $s_{2,1} = 4391.40 < 4800 = B$ and

$$V_{2,3} = e^{-\log(1+0.04)\cdot 2/52}(0.22 \cdot 606.07 + 0.55 \cdot 90 + 0.23 \cdot 0) = 184.33$$

(see Figure 17). We can continue further and compute the option price at times $t_1 = 2$ and $t_0 = 0$ just using the standard formula (28) since prices no longer lie below the barrier B (see Figure 14). Thus, one computes $V_{1,1} = 79.7$, $V_{1,2} = 251.7$, $V_{1,3} = 639.8$, and finally, the option price at time $t_0 = 0$ and stock price $S = 5290$ equals

$$V_{0,1} = e^{-\log(1+0.04)\cdot 2/52}(0.25 \cdot 639.8 + 0.50 \cdot 251.7 + 0.25 \cdot 79.7) = 303.28.$$

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